AMGM Refined? Or Is It?

The lovely new book Borwein, Bailey, and Gergensohn (2004, p. 210) quotes a nice problem: Show that for all n and all nonnegative a_k, $1 \le k \le n$, one has the bound

$$1 + (a_1 a_2 \cdots a_n)^{1/n} \le \prod_{1 \le i \le n} (1 + a_i)^{1/n} \qquad (1)$$

One thing to note about this inequality is that by the AMGM inequality the right side is bounded above by

$$1 + \frac{1}{n} \sum_{1 \le i \le n} a_i,$$

so after canceling the 1's we get the AMGM inequality. Consequently, there is precise sense in which the problem bound (1) refines the AMGM inequality. Still, there is a hint of paradox here, as one can see by the following proof of the problem bound (1).

A Proof

We divide the left side of (1) by the right side and apply the AMGM inequality twice. This gives gives us

$$\frac{1}{\prod_{1 \le i \le n}(1 + a_i)^{1/n}} + \frac{(a_1 a_2 \cdots a_n)^{1/n}}{\prod_{1 \le i \le n}(1 + a_i)^{1/n}}$$
$$\le \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 + a_i} + \frac{1}{n} \sum_{i=1}^{n} \frac{a_i}{1 + a_i} = 1,$$

so the problem inequality is proved.

Paradoxical to You?

I flip back and forth trying to decide if these two arguments are paradoxical. The bound (1) implies the AMGM inequality and the AMGM inequality implies the bound (1), so the two are clearly equivalent as mathematical assertions. But, hey, not so fast, doesn't the problem bound (1) also tell us just a little bit more in some circumstances? Should we let mere mathematical equivalence tell us which of our tools is sharper? Let me know what you think.

More to Say?

This problem fits nicely into a somewhat larger context. Specifically, Minkowski observed that the geometric mean $G(a_1, a_2, ..., a_n)$ is superadditive in the sense that

$$G(a_1, a_2, ..., a_n) + G(b_1, b_2, ..., b_n) \le G(a_1 + b_1, a_2 + b_2, ..., a_n + b_n).$$

Naturally, Minkowski's bound covers the problem inequality if you take $b_k = 1$ for all k. There's a bit more about this in (The Cauchy-Schwarz Master Class, Chapter 2).

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